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Commutative Algebra
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Exercise sheet 1

Ex 1. Let R be a ring and let R' be an R-algebra. Let $\mu\colon R' \otimes_R R' \to R'$ be the multiplication map, $x \otimes y \mapsto xy$. Note that μ is not injective, and write $I_{R'}$ for $\ker(\mu)$. Show that $I_{R'}$ is generated by the elements of the form $x \otimes 1 - 1 \otimes x$, where x runs on the set of generators of R' as R algebra.

Let now B be an R algebra, and let B', B'' be two B-algebras. Set $C = B' \otimes_R B''$. Prove that 1) $C/I_B C \cong B' \otimes_B B''$ and 2) $B' \otimes_B B'' \cong B \otimes_{B\otimes_R B} C$. For the latter isomorphism, we consider B as $B \otimes_R B$-algebra via the multiplication map μ.

Ex 2. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = 0$ if n, m are coprime.

Ex 3. Let $K = \mathbb{Q}[\sqrt[5]{2}]$. Compute $K \otimes_{\mathbb{Q}} K$.

Ex 4. Let R be a ring, R' an R-algebra and M an R'-module. Write $M' = M \otimes_R R'$. Show that M is a direct factor of M' (construct a morphism from M to M' and from M' to M, and show that the composition $M \to M' \to M$ is the identity).

Ex 5. Let R be a local ring, and let M, N be finitely generated R-modules. Prove that if $M \otimes_R N = 0$, then either $M = 0$ or $N = 0$. (Hint: reduce to the case of fields and use Nakayama).

Ex 6. Let R be a ring and let A and B be R-algebras. Write $\mathbf{Alg}_R$ for the category of R-algebras. Consider the functor $F\colon \mathbf{Alg}_R \to \mathbf{Set}$ given on objects by

$$C \mapsto \hom_R(A, C) \times \hom_R(B, C)$$

and on morphisms in the obvious way by post-composition. Show that it is representable, and that the representable object is precisely $A \otimes_R B$ as R-algebra.

Ex 7. Let A be an integral domain. An A-module M is called *torsion free* if for any $0 \neq a \in A$ the multiplication map $\mu_a\colon M \to M$, $m \mapsto am$ is injective. Show that being torsion free is a local property for M, i.e. that the following are equivalent:

i) M is torsion free
ii) $M_{\mathfrak{p}}$ is torsion free for every prime ideal $\mathfrak{p} \subset A$
iii) $M_{\mathfrak{m}}$ is torsion free for every maximal ideal $\mathfrak{m} \subset A$.

Ex 8. Let A be a ring. Show that being reduced for A is a local property.

Ex 9. Let A be a ring and let S be a multiplicatively closed subset of A. Let M be a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sm = 0$ for every $m \in M$.

Ex 10. Let A be a ring, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Assume that M'' is finitely generated and flat. Show that the sequence

$$0 \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is exact for every A-module N. (Hint: by choosing a set of generators, write a presentation of M'' as $0 \to K \to A^n \to M'' \to 0$. Then tensor this sequence with the given sequence and contemplate the resulting diagram. Use the fact that A^n is free, hence flat A-module and the snake lemma to conclude).